

## BASIC EQUATIONS OF ELECTRODYNAMICS

### Maxwell's Equations

*In general:*

$$\left\{ \begin{array}{l} \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{array} \right.$$

### Auxiliary Fields

*Definitions:*

$$\left\{ \begin{array}{l} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \end{array} \right. \quad \left\{ \begin{array}{l} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E} \\ \mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{H} = \frac{1}{\mu} \mathbf{B} \end{array} \right.$$

### Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}$$

### Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

### Energy, Momentum, and Power

*Energy:* 
$$U = \frac{1}{2} \int (\epsilon_0 E^2 + \frac{1}{\mu_0} B^2) d\tau$$

*Momentum:* 
$$\mathbf{P} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B}) d\tau$$

*Poynting vector:* 
$$\mathbf{S} = \frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B})$$

*Larmor formula:* 
$$P = \frac{\mu_0}{6\pi c} \dot{q}^2 a^2$$

*In matter:*

$$\left\{ \begin{array}{l} \nabla \cdot \mathbf{D} = \rho_f \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \end{array} \right.$$

*Linear media:*

$$\left\{ \begin{array}{l} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E} \\ \mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{H} = \frac{1}{\mu} \mathbf{B} \end{array} \right.$$

## FUNDAMENTAL CONSTANTS

$\epsilon_0 = 8.85 \times 10^{-12} \text{ C}^2/\text{Nm}^2$	(permittivity of free space)
$\mu_0 = 4\pi \times 10^{-7} \text{ N/A}^2$	(permeability of free space)
$c = 3.00 \times 10^8 \text{ m/s}$	(speed of light)
$e = 1.60 \times 10^{-19} \text{ C}$	(charge of the electron)
$m = 9.11 \times 10^{-31} \text{ kg}$	(mass of the electron)

## SPHERICAL AND CYLINDRICAL COORDINATES

### Spherical

$$\left\{ \begin{array}{l} x = r \sin \theta \cos \phi \\ y = r \sin \theta \sin \phi \\ z = r \cos \theta \end{array} \right. \quad \left\{ \begin{array}{l} \hat{x} = \sin \theta \cos \phi \hat{r} + \cos \theta \cos \phi \hat{\theta} - \sin \phi \hat{\phi} \\ \hat{y} = \sin \theta \sin \phi \hat{r} + \cos \theta \sin \phi \hat{\theta} + \cos \phi \hat{\phi} \\ \hat{z} = \cos \theta \hat{r} - \sin \theta \hat{\theta} \end{array} \right.$$

$$\left\{ \begin{array}{l} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}(\sqrt{x^2 + y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{array} \right.$$

### Cylindrical

$$\left\{ \begin{array}{l} x = s \cos \phi \\ y = s \sin \phi \\ z = z \end{array} \right. \quad \left\{ \begin{array}{l} \hat{x} = \cos \phi \hat{s} - \sin \phi \hat{\phi} \\ \hat{y} = \sin \phi \hat{s} + \cos \phi \hat{\phi} \\ \hat{z} = \hat{z} \end{array} \right.$$

$$\left\{ \begin{array}{l} s = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{array} \right. \quad \left\{ \begin{array}{l} \hat{s} = \cos \phi \hat{x} + \sin \phi \hat{y} \\ \hat{\phi} = -\sin \phi \hat{x} + \cos \phi \hat{y} \\ \hat{z} = \hat{z} \end{array} \right.$$

**Cartesian.**  $d\mathbf{l} = dx \hat{x} + dy \hat{y} + dz \hat{z}; \quad d\tau = dx dy dz$

**Gradient:**  $\nabla t = \frac{\partial t}{\partial x} \hat{x} + \frac{\partial t}{\partial y} \hat{y} + \frac{\partial t}{\partial z} \hat{z}$

**Divergence:**  $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

**Curl:**  $\nabla \times \mathbf{v} = \left( \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{x} + \left( \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{y} + \left( \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{z}$

**Laplacian:**  $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

**Spherical.**  $d\mathbf{l} = dr \hat{r} + r d\theta \hat{\theta} + r \sin\theta d\phi \hat{\phi}; \quad d\tau = r^2 \sin\theta dr d\theta d\phi$

**Gradient:**  $\nabla t = \frac{\partial t}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial t}{\partial \theta} \hat{\theta} + \frac{1}{r \sin\theta} \frac{\partial t}{\partial \phi} \hat{\phi}$

**Divergence:**  $\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta v_\theta) + \frac{1}{r \sin\theta} \frac{\partial v_\phi}{\partial \phi}$

**Curl:**  $\nabla \times \mathbf{v} = \frac{1}{r \sin\theta} \left[ \frac{\partial}{\partial \theta} (\sin\theta v_\phi) - \frac{\partial v_\theta}{\partial \phi} \right] \hat{r}$

$+\frac{1}{r} \left[ \frac{1}{\sin\theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r} (r v_\phi) \right] \hat{\theta} + \frac{1}{r} \left[ \frac{\partial}{\partial r} (r v_\theta) - \frac{\partial v_r}{\partial \theta} \right] \hat{\phi}$

**Laplacian:**  $\nabla^2 t = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial t}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left( \sin\theta \frac{\partial t}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 t}{\partial \phi^2}$

**Cylindrical.**  $d\mathbf{l} = ds \hat{s} + s d\phi \hat{\phi} + dz \hat{z}; \quad d\tau = s ds d\phi dz$

**Gradient:**  $\nabla t = \frac{\partial t}{\partial s} \hat{s} + \frac{1}{s} \frac{\partial t}{\partial \phi} \hat{\phi} + \frac{\partial t}{\partial z} \hat{z}$

**Divergence:**  $\nabla \cdot \mathbf{v} = \frac{1}{s} \frac{\partial}{\partial s} (s v_s) + \frac{1}{s} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$

**Curl:**  $\nabla \times \mathbf{v} = \left[ \frac{1}{s} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right] \hat{s} + \left[ \frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s} \right] \hat{\phi} + \frac{1}{s} \left[ \frac{\partial}{\partial s} (s v_\phi) - \frac{\partial v_s}{\partial \phi} \right] \hat{z}$

**Laplacian:**  $\nabla^2 t = \frac{1}{s} \frac{\partial}{\partial s} \left( s \frac{\partial t}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$

VECTOR IDENTITIES

Triple Products

(1)  $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2)  $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3)  $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4)  $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5)  $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6)  $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7)  $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8)  $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9)  $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10)  $\nabla \times (\nabla f) = 0$

(11)  $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

FUNDAMENTAL THEOREMS

**Gradient Theorem:**  $\int_a^b (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

**Divergence Theorem:**  $\int (\nabla \cdot \mathbf{A}) d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

**Curl Theorem:**  $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$